Improved coupled Z–R and k–R relations and the resulting ambiguities in the determination of the vertical distribution of rain from the radar backscatter and the integrated attenuation

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Abstract

Several algorithms to calculate a rain-rate profile from a single-frequency air- or space-borne radar backscatter profile and a given path-integrated attenuation have been proposed, The accuracy of any such algorithm is limited by the ambiguities between the (multiple) exact solutions, which depend on the variability of the parameters in the Z--R and k–R relations used. In this study, we derive coupled Z-R and k–R relations based on the drop-size-distribution. We then show that, because of the coupling, the relative difference between the multiple mutually ambiguous rain rate profiles solving the problem must remain acceptably low, provided the available path-integrated attenuation value is known to within 0.5 dB.

1 Introduction

The problem of estimating the vertical rain profile from measurements obtained using a single-frequency active air- or space-borne radar, and subject to a path-integrated attenuation condition, has already been studied. Indeed, several algorithms have been proposed to find a "solution", i.e. a rain rate profile that does the job. (see for example Kozu and Nakamura 1991, Meneghini and Nakamura 1990, Weinman et al 1990). However, even under ideal conditions where the measurements are perfectly calibrated, and in the absence of noise, the problem has multiple solutions. As is shown in (Haddad and Im 1993, and Haddad et al 1994), this ambiguity is due to the fact that the parameters governing the classical power laws relating the radar reflectivity coefficient Z and the radar attenuation coefficient k to the rain rate R (or liquid water content W) depend on the drop-size distribution. For example, in the Z-R relation Z = aR^b, empirical measurements have shown substantial variations in a and b. In fact, Battan, 1973, reports over 70 Z–R relations due to differing drop size distributions. Without a path integration constraint, rain retrieval requires assumed values for the parameters a, b, and the corresponding parameters in the k–R relation. Each combination of parameters provides a different solution. The path attenuation constraint provides one additional equation relating the four parameters. However, without further constraints, because of the variability of these parameters, substantially different rain profiles can still be generated that produce the measured power profile and path-integrated attenuation (Haddad and Irn, 1993).

As was pointed out in (Haddad et al, 1994), this ambiguity can be reduced if improved Z-R and k–R relations are used, specifically ones that do account realistically for the inter-dependence of the various parameters involved. In this paper, wc derive such Z–1? and k–R relations at Ku-band using realistic scattering models and the results of Jameson, 1993 and 1994. Ku-band is of particular interest because it is the band of the Tropical Rainfall Measuring Mission's precipitation radar (Nakamura et al, 1990). We also derive the exact expressions for the mutually ambiguous rain profiles, and show that while these multiple solutions are indeed mathematically different, the difference is relatively small if the path-integrated attenuation is known exactly. When the latter can be determined only to within a known uncertainty, we also show how our formulas produce bounds on the relative difference between mutually ambiguous profiles as a function of the relative error in the path-integrated attenuation.

In section 2, we briefly describe the ambiguity problem, and reproduce the analytic formulas giving all the mutually ambiguous rain profiles that produce the same given radar reflectivity profile and the same given path-integrated attenuation. The details can be found in (Haddad et al, 1994). In section 3, we describe our improved Z–R and k-n relations and derive the corresponding formula for the relative difference between two mutually ambiguous profiles. In section 4 wc show how these formulas allows us to conclude that a perfect knowledge of the path-integrated attenuation makes the ambiguities acceptably small,

2 The ambiguity formula

In this section, we shall write down the exact formula relating all the mutually ambiguous profiles to a single "reference" profile, assuming that our data consist of the direct radar reflectivity profile along with the path-integrated attenuation. To retain some generality, we shall assume that the "path-integrated attenuation" measurement may contain some error. For example, it may be obtained from the backscatter from that portion of the surface that underlies the rain,

Mathematically, the backscattered radar power from range r is proportional to the reflectivity coefficient Z(r) of the rain at range r, and to the accumulated attenuation from range O to range r. Calling $k(r)$ (resp. $R(r)$) the attenuation coefficient (resp. rain rate) at range r, we begin with the simple empirical model that $Z = aR^b$ and $k = \alpha R^\beta$ for some value of the parameters a, b, α and β, and that the backscattered power is therefore given by

$$p(r) = C(r) \, . \, aR(r)^b 10^{-0.2\alpha \int_0^r R(t)^\beta dt}, \tag{1}$$

where r = O is the range at the top of the cloud, $C(r)$ represents the range-dependent calibration constant, which we assume to be known exactly. Since it is highly unlikely that a, b, α or β are ever known exactly, we would like to quantify the effect on R of an error in these parameters.

To fix the notation, let us call $\{a_0, b_0, \alpha_0, \beta_0\}$ our reference set of rain parameters giving rise to the rain rate profile $R_0(r)$, and determine under what conditions a different set of parameters $\{a_1, b_1, \alpha_1, \beta_1\}$ with rain-rate profile $R_1(r)$ can still give rise to the same backscattered power $p(r)$, and to the same path-integrated attenuation $10^{-0.2\alpha \int_0^{r_s} R(t)^\beta dt} \, . \, \chi$, where r_s is the distance from the top of the cloud to the surface, and χ denotes the multiplicative error in the path-integrated attenuation value. This error could be due to an error in the surface backscattering coefficient, in the case where the surface echo is used to estimate the path-integrated attenuation, or to calibration and modeling errors, in the case where the integrated attenuation is inferred from a low-frequency passive microwave brightness temperature. The more special case where the exact path-integrated attenuation is' assumed known can be obtained from the more general situation considered here by assuming χ to be exactly equal to 1.

Thus, we are given $\{R_0(r), a_0, b_0, \alpha_0, \beta_0, \chi_0\}$, and we must determine under what conditions a different $\{R_1(r), a_1, b_1, \alpha_1, \beta_1, \chi_1\}$ could still satisfy

$$aR_1(r)^b 10^{-0.2\alpha_1 \int_0^r R_1(t)^\beta dt} = aR_0(r)^b 10^{-0.2\alpha_0 \int_0^r R_0(t)^\beta dt}, \tag{2}$$

along with

$$10^{-0.2\alpha_1 \int_0^{r_s} R_1(t)^{\beta} dt} \cdot \chi_1 = 10^{-0.2\alpha_0 \int_0^{r_s} R_0(t)^{\beta} dt} \cdot \chi_0. \tag{3}$$

At first, solving equations (2) and (3) simultaneously seems quite daunting. We start by noting that the requirement that equations (2) and (3) be satisfied is equivalent to requiring that $p_0(r) = p_1(r)$ and that

$$\frac{p_0(r)}{10^{-0.2\int_0^{r_s} k_0(t)dt} \cdot \chi_0} = \frac{p_1(r)}{10^{-0.2\int_0^{r_s} k_1(t)dt} \cdot \chi_1} \tag{4}$$

at all ranges r. This observation greatly simplifies the problem of finding the multiple solutions. Indeed, we had already shown in (Haddad and Im, 1993) that these last two conditions force our rainrate R_1 to satisfy the two equations

$$R_1(r) = \frac{R_0(r)^{b_0/b_1} 10^{-0.2\frac{\alpha_0}{b_1}\int_0^r R_0(t)^{\beta_0} dt}}{\left((a_1/a_0)^{\beta_1/b_1} - \frac{0.2\log(10)\alpha_1\beta_1}{b_1}\int_0^r R_0(r')^{b_0\beta_1/b_1} 10^{-0.2\frac{\alpha_0\beta_1}{b_1}\int_0^{r'} R_0(t)^{\beta_0} dt} dr'\right)^{1/\beta_1}} \tag{5}$$

and

$$R_1(r) = \frac{R_0(r)^{b_0/b_1} 10^{0.2\frac{\alpha_0}{b_1}\int_r^{r_s} R_0(t)^{\beta_0} dt}}{\left(\left(\frac{\chi_0 a_1}{\chi_1 a_0}\right)^{\beta_1/b_1} + \frac{0.2\log(10)\alpha_1\beta_1}{b_1}\int_r^{r_s} R_0(r')^{b_0\beta_1/b_1} 10^{0.2\frac{\alpha_0\beta_1}{b_1}\int_{r'}^{r_s} R_0(t)^{\beta_0} dt} dr'\right)^{1/\beta_1}}. \tag{6}$$

in (Haddad et al, 1994), we then showed how one can solve the system consisting of these two equations simultaneously. The solution is given by

$$R_1(r) = \frac{R_0(r)^{b_0/b_1} 10^{-0.2\frac{\alpha_0}{b_1}\int_0^r R_0(t)^{\beta_0} dt}}{\left((a_1/a_0)^{\beta_1/b_1} - \frac{0.2\log(10)\alpha_1\beta_1}{b_1}\int_0^r R_0(r')^{b_0\beta_1/b_1} 10^{-0.2\frac{\alpha_0\beta_1}{b_1}\int_0^{r'} R_0(t)^{\beta_0} dt} dr'\right)^{1/\beta_1}} \tag{5}$$

and

$$\left(\frac{a_0}{a_1}\right)^{\beta_1/b_1} \cdot \frac{\alpha_1}{\alpha_0} = \frac{A^{\beta_1/b_1} - \rho^{\beta_1/b_1}}{A^{\beta_1/b_1} \cdot R_0(0)^{(\frac{\beta_1}{b_1} - \frac{\beta_0}{b_0})b_0} - R_0(r_s)^{(\frac{\beta_1}{b_1} - \frac{\beta_0}{b_0})b_0} + (\frac{\beta_1}{b_1} - \frac{\beta_0}{b_0})b_0 \cdot I} \tag{7}$$

where A denotes the 2-way path-integrated attenuation, $A = 10^{0.2\alpha_0\int_0^{r_s} R_0(t)^{\beta_0} dt}$, $I(r)$ is the integral $I(r) = \int_0^r R_0'(r') R_0(r')^{(\frac{b_0}{b_1}\beta_1 - \beta_0 - 1)} 10^{0.2\frac{\alpha_0\beta_1}{b_1}\int_{r'}^{r_s} R_0(t)^{\beta_0} dt} dr'$, and ρ is the ratio of the uncertainties in the path-integrated attenuation, $\rho = \chi_0/\chi_1$. Thus, in order for $R_1(r)$, a_1, b_1, α_1 and β_1 to produce the same radar backscatter and the same path-integrated attenuation as $R_0(r), a_0, b_0, \alpha_0$ and β_0, the parameters a_1, b_1, α_1 and β_1 must satisfy (7), and R_1 must satisfy (5).

In order to derive a bound on the relative difference $R_1/R_0 - 1$, we rewrite equation (5) as

$$R_1(r) = \left(\frac{\frac{\alpha_0}{\alpha_1} R_0(r)^{\beta_1 b_0/b_1} 10^{-0.2\frac{\beta_1}{b_1}\int_0^r \alpha_0 R_0(t)^{\beta_0} dt}}{\left(\frac{\alpha_0}{\alpha_1}\frac{a_1}{a_0}\right)^{\beta_1/b_1} + \int_0^r R_0(r')^{\nu} \left(-0.2 \log(10) \frac{\beta_1}{b_1} \alpha_0 R_0(r')^{\beta_0} 10^{-0.2\frac{\beta_1}{b_1}\int_0^{r'} \alpha_0 R_0(t)^{\beta_0} dt}\right) dr'} \right)^{1/\beta_1} \tag{8}$$

where wc have called v the expression $\nu = (\beta_1/b_1 - \beta_0/b_0)b_0$. We then perform the integration in the denominator by parts, then multiply the numerator and denominator by $10^{0.2\frac{\beta_1}{b_1}\int_0^r \alpha_0 R_0(t)^{\beta_0} dt}$. Our equation becomes

$$R_1(r) = \left(\frac{\frac{\alpha_0}{\alpha_1} R_0(r)^{\beta_1 b_0/b_1}}{\left(\frac{\alpha_0}{\alpha_1}\left(\frac{a_1}{a_0}\right)^{\beta_1/b_1} - R_0(0)^{\nu}\right) 10^{0.2\frac{\beta_1}{b_1}\int_0^r \alpha_0 R_0(t)^{\beta_0} dt} + R_0(r)^{\nu} - \epsilon(r)} \right)^{1/\beta_1} \tag{9}$$

where we have used the abbreviation

$$\epsilon(r) = \nu \int_0^r \frac{R_0'(r')}{R_0(r')} R_0(r')^{\nu} 10^{0.2\frac{\beta_1}{b_1}\int_{r'}^r \alpha_0 R_0(t)^{\beta_0} dt} dr' \tag{lo}$$

Similarly, we rewrite the path-integrated-attenuation constraint (7) as

$$\frac{\alpha_0}{\alpha_1}\left(\frac{a_1}{a_0}\right)^{\beta_1/b_1} = \frac{A^{\beta_1/b_1} \cdot R_0(0)^{(\frac{\beta_1}{b_1} - \frac{\beta_0}{b_0})b_0} - R_0(r_s)^{(\frac{\beta_1}{b_1} - \frac{\beta_0}{b_0})b_0} + \epsilon(r_s)}{A^{\beta_1/b_1} - \rho^{\beta_1/b_1}} \tag{11}$$

In the appendix, we show that, because the ratio β/b cannot vary significantly, ν must remain small, and therefore ϵ is small enough that it may (and will) be neglected in both equations. With this simplification, and after we substitute the right-hand-side of (11) for the expression $(\alpha_0/\alpha_1)\cdot(a_1/a_0)^{\beta_1/b_1}$ *in (9),* the latter becomes

$$R_1(r) = \left(\frac{\alpha_0}{\alpha_1}\right)^{1/\beta_1} R_0(r)^{\beta_0/\beta_1} \left(\frac{R_0(r)^{\nu}}{\left(\frac{R_0(0)^{\nu}\rho^{\beta_1/b_1} - R_0(r_s)^{\nu}}{A^{\beta_1/b_1} - \rho^{\beta_1/b_1}}\right) 10^{0.2\frac{\beta_1}{b_1}\int_0^r \alpha_0 R_0(t)^{\beta_0} dt} + R_0(r)^{\nu}} \right)^{1/\beta_1} \tag{12}$$

$$= \frac{(\alpha_0/\alpha_1)^{1/\beta_1} R_0(r)^{\beta_0/\beta_1}}{\left(\left(\frac{(R_0(0)/R_0(r))^{\nu}\rho^{\beta_1/b_1} - (R_0(r_s)/R_0(r))^{\nu}}{A^{\beta_1/b_1} - \rho^{\beta_1/b_1}}\right) 10^{0.2\frac{\beta_1}{b_1}\int_0^r \alpha_0 R_0(t)^{\beta_0} dt} + 1\right)^{1/\beta_1}} \tag{13}$$

Equation (13) is the relation between R_1 and R_0 that we will use in section 4 to derive an upper bound on the ambiguity. Wc conclude this section by noting that several studies have sought to determine realistic values for the parameters a, b, α and β from measurements and from more or less sophisticated scattering models (see, for example, Battan 1973, Doviak and Zrnić 1984, Ulbrich 1983). Almost all published values fall within the ranges

$$200 \leq a \leq 600 \tag{14}$$
$$1.2 \leq b \leq 1.6 \tag{15}$$
$$0.017 < \alpha < 0.034 \tag{16}$$
$$1 \leq \beta \leq 1.2 \tag{17}$$

when R is expressed in mm/hr, Z in $mm^6/m3$ k in dB/km. These constraints are not tight enough to allow us to infer that all rain-rate profiles R_1 satisfying (5) and (7) must be close to R_0. In fact, as is shown in (Haddad et al, 1994), the relative difference between R_0 and R_1 can easily exceed 100%. It is however natural to suspect that the inequalities (14,1 5,16,17) are not the only constraints governing the values of a, b, α and β. For example, one knows from Rayleigh models that the coefficients should be proportional to some power of the mean drop diameter: since higher rain rates correspond to larger drop sizes, one would suspect that the coefficients a and α must somehow "grow together" at higher rain rates, and similarly "shrink together" at lower rain rates. One would expect a similar approximate behavior of the ratio b/β. In the next section, we shall show that one can indeed derive expressions for the coefficients a, b, α and β that exhibit their correlation more readily. When substituted back in equations (5) and (7), these improved expressions for our parameters will allow us to obtain sufficiently tight bounds on the relative difference between R_0 and RI.

3 Improved coupled radar↔rain relations

In (Jameson 1993 and 1994), it was shown that the attenuation could be paramterized using the mass-weighted mean drop diameter $\overline{D}$ and R (or W), i.e. that a single relationship between k, R and $\overline{D}$ holds for a large class of drop size distributions. Figure 1 reproduces the scatter diagram of k/R versus $\overline{D}$ from (Jameson, 1994). These values were obtained with R between 1 and 200 mm/hr. In addition, a Γ drop-size distribution $N(D) = N_0 D^{\mu} e^{-\Lambda D}\, mm^{-1} m^{-3}$ as in (Ulbrich, 1983) was assumed, with parameters A and μ varying independently over the intervals 1,6 $\leq$ A $\leq$ 4.5 mm-l and $-1 < \mu \leq 2$ respectively. The parameter NO is then directly related to R, A and μ by the formula No $\simeq$ 141. $R \,.\, \Lambda^{\mu+4.67}/\Gamma(\mu + 4.67)$, if we assume that the drops are falling at their terminal drop velocity $v(D)$ given by $v(D) = 3.78 D^{0.67}$ m/see. The scattering calculations were made assuming a frequency of 13.8 GHz and with temperatures between O"C and 20° C. A T-matrix algorithm was used, with realistically-shaped scatterers. These calculations produced a large data set consisting of quadruples (R, $\overline{D}, k, Z$) for the different values of A, μ and temperatures above.

Figure 1 suggests that one could write the *k-R* relation in the form

$$k = \lambda \overline{D} R \,, \tag{18}$$

although we may have to allow the parameter λ to vary over a relatively wide interval. To minimize the uncertainty in λ, instead of (18), we look for those exponents 1 and *n* that will allow us to write

$$k = C_1 \overline{D}^l R^n \tag{19}$$

with an inequality for the coefficient C_1

$$C_1^{min} \leq C_1 \leq C_1^{max} \tag{20}$$

that will be valid for all our data points, and for which the ratio C_1^{max}/C_1^{min} is as small as possible. Using our calculated data $\{(R_j, \overline{D}_j, k_j, Z_j)\}$ (the subscript j denoting the j^{th} data point), we computed the ratio

$$\frac{\max_j \left(k_j/(\overline{D}_j^l R_j^n)\right)}{\min_j \left(k_j/(\overline{D}_j^l R_j^n)\right)} \tag{21}$$

for l and *n* in the intervals [0.5, 1.5], and chose that pair of vlaues for which the ration is minimized. It turned out that the optimal values are 1 = 0,86 and n = 1, for which we can write

$$k = C_1 \overline{D}^{0.86} R \,,\ 0.0222 < \ C, \ < 0.026 \tag{22}$$

In the case of the reflectivity coefficient Z, one could intuitively guess that the ratio Z/R should be more closely proportional to $\overline{D}^3$. Figure 2 shows the computed scatter diagram of *Z/R* versus $\overline{D}$, under the same hypotheses listed above. To obtain a *Z–R* relation with coefficients that are as tightly constrained as possible, one proceeds as in the case of *k,* looking for those exponents 1 and *n* for which the quantity

$$\frac{\mathrm{maXj} \left(Z_j/(\overline{D}_j^l R_j^n)\right)}{\min_j \left(Z_j/(\overline{D}_j^l R_j^n)\right)} \tag{23}$$

is minimized. It turns out in this case that l = 2,59 and n = 1, with the resulting relation

$$z = C_2 \overline{D}^{2.59} R, \ \ 203 < \ C_2 \ < 470 \tag{24}$$

Equations *(22)* and *(24)* by themselves are still not sufficient for our purposes. Indeed, they both involve the parameter $\overline{D}$. Since $\overline{D}$ and R are not independent, we must either "eliminate" $\overline{D}$ between (22) and (24), or make the correlation between $\overline{D}$ and *R* explicit, then "try to derive appropriate constraints on the coefficients of the resulting $\overline{D}$–R relation. Since the latter option produces tighter constraints on the coefficients of the k–R and Z–R relations, we shall choose it and write $\overline{D} = \gamma R^{\delta}$. There remains to determine the range of values over which γ and δ can realistically vary. To that end, note that our k–R and Z–R relations can now be written as

$$k = C_1 \gamma^{0.86} R^{1+0.86\delta} \quad , \quad 0.0222 < C_1 < 0.026 \tag{25}$$
$$Z = C_2 \gamma^{2.59} R^{1+2.59\delta} \quad , \quad 203 < C_2 < 470 \tag{26}$$

All we have at our disposal to derive realistic constraints on the $\overline{D}$–R coefficients γ and δ are the empirical bounds (14, 15, 16, 17). Indeed, (25) and (26) imply that our new parameters Cl, C_2, γ and δ are related to the original power-law parameters a, b, α and β by

$$a = C_2 \gamma^{2.59} \, , \tag{27}$$
$$b = 1 + 2.59\delta \, , \tag{28}$$
$$\alpha = C_1 \gamma^{0.86} \, , \tag{29}$$
$$\beta = 1 + 0.866. \tag{30}$$

These relations allow us to translate the constraints (14, 15, 16, 17) directly into bounds on γ and δ, namely

$$200 < C_{2,min} \gamma_{min}^{2.59} \quad \text{and} \quad C_{2,max} \gamma_{max}^{2.59} < 600, \tag{31}$$
$$1.2 < 1 + 2.59\delta_{min} \quad \text{and} \quad 1 + 2.59\delta_{max} < 1.6 \, , \tag{32}$$
$$0.017 < C_{1,min} \gamma_{min}^{0.86} \quad \text{and} \quad C_{1,max} \gamma_{max}^{0.86} < 0.034, \tag{33}$$
$$1 < 1 + 0.86\delta_{min} \quad \text{and} \quad 1 + 0.86\delta_{max} < 1.2, \tag{34}$$

from which we conclude that γ and δ must remain within the bounds

$$0.99 < \gamma < 1.1, \tag{35}$$
$$0.08 < \delta < 0.23. \tag{36}$$

Since they have been derived from an eclectic assortment of k–R and Z-R relations obtained by different scientists using different methods, the bounds in constraints (35) and (36) are undoubtedly not as sharp as they could be made if one conducted a statistical analysis of carefully collected representative samples of drop-size distributions. Yet, as we shall see in the following section, these bounds are sharp enough to allow us to infer a satisfactorily low bound on the ambiguity in the rain profile determination problem, This is due in no small part to the fact that (25) and (26) make the interdependence between the parameters of the k–R relation and those of the Z–R relation explicit. These correlations between the various a priori unknown coefficients in effect eliminates large families of combinations of values that would otherwise have allowed significantly different rain rates to produce the same radar backscatter and the same path-integrated attenuation.

4 Bounds on the ambiguity

We now re-examine equations (5) and (7), in light of constraints (25), (26), (35) and (36), in order to quantitatively estimate the ambiguity, i.e. the difference between any rain-rates R_1 which satisfy (5) and (7), and the reference rain-rate R_0, where R_0 corresponds to a set of reference values of the rain parameters. In this discussion, we will only need the actual values of α_0 and β_0. We shall therefore assume that the value&= 0.15 was used to obtain R_0, which implies that β_0 = 1.1185, and we further choose α_0 at the lower end of the range given by equations (18) and (28), say α_0 = 0.022, Since it is not reasonable to expect that the RI's will be close to R_0 for all values large and small, from a fraction of a millimeter per hour up to hundreds of mm/hr, we shall also restrict our attention to rain rate values that fall within a pre-specified interval

$$R_{min} \leq R_0(r) \leq R_{max} \tag{37}$$

We first **assume that the path-integrated attenuation is known exactly,** i.e. that ρ = 1, and look for bounds on the denominator $\mathcal{D}(r)$ of equation (13). With ρ = 1, $\mathcal{D}(r)$ can be rewritten as

$$\mathcal{D}(r) = \left[1 + \left(\left(\frac{R_0(0)}{R_0(r)}\right)^{\nu} - \left(\frac{R_0(r_s)}{R_0(r)}\right)^{\nu} \right) \cdot \frac{\left(10^{0.2\int_0^r \alpha_0 R_0(t)^{\beta_0} dt}\right)^{\beta_1/b_1}}{A^{\beta_1/b_1} - 1} \right]^{1/\beta_1} \tag{38}$$

Over the range of rain rates given by (37), equation (38) implies that

$$\left[1 + \left(1 - \left(\frac{R_{max}}{R_{min}}\right)^{\nu}\right) \cdot \frac{A^{\beta_1/b_1}}{A^{\beta_1/b_1} - 1}\right]^{1/\beta_1} \leq \mathcal{D}(r) \leq \left[1 + \left(\left(\frac{R_{max}}{R_{min}}\right)^{\nu} - 1\right) \cdot \frac{A^{\beta_1/b_1}}{A^{\beta_1/b_1} - 1}\right]^{1/\beta_1} \tag{39}$$

We now restrict our attention to cases where the path-integrated attenuation is large enough to be significant, say where it is at least 2 dB (with a nominal value of β = 1.1 and with α between 0.022 and. 0.028, this would correspond to a rain rate between 14 and 17 mm/hr over 4 km), In that case, and because β/b is always greater than 0.74, we know that $A^{\beta_1/b_1} \geq$ 1.98, so that $A^{\beta_1/b_1}/(A^{\beta_1/b_1} - 1) \leq 2$. Also, since δ_0 = 0.15, the exponent ν is bounded by $-0.09 \leq \nu \leq 0.1$. Together,. these inequalities allow us to rewrite (39) as

$$\left[1 - 2\left(\left(\frac{R_{max}}{R_{min}}\right)^{0.1} - 1\right)\right]^{1/\beta_1} \leq \mathcal{D}(r) \leq \left[1 + 2\left(\left(\frac{R_{max}}{R_{min}}\right)^{0.1} - 1\right)\right]^{1/\beta_1} \tag{40}$$

Thus, if we are interested in a range of values R_{max}/R_{min} = 4, (13) and (40) imply that

$$R_1(r) = \left(\frac{\alpha_0}{\alpha_1}\right)^{1/\beta_1} R_0(r)^{\beta_0/\beta_1} \frac{1}{\mathcal{D}} \quad \text{with} \quad 0.78 \leq \frac{1}{\mathcal{D}} \leq 1.4 \tag{41}$$

Since, in addition, $0.7 \leq (\alpha_0/\alpha_1)^{1/\beta_1} \leq 1$, we can finally write

$$0.55 R_0(r)^{\beta_0/\beta_1 - 1} - 1 \leq \frac{R_1(r)}{R_0(r)} - 1 \leq 1.4 R_0(r)^{\beta_0/\beta_1 - 1} - 1 \tag{42}$$

We know that with δ_0 = 0.15, $|\beta_0/\beta_1 - 1|$ cannot exceed 0.05. Even when R_{max} = 50 mm/hr, the term $R_0(r)^{\beta_0/\beta_1 - 1}$ can therefore not exceed 1,21 . Thus (42) allows us to conclude that the extreme relative difference between R_1 and R_0 will not exceed 70'%0.

The 70%-estimate was obtained assuming that the path-integrated attenuation could be as small as 2 dB. As the attenuation increases, (39) shows that our bound on the relative error should decrease. Figure 3 shows the plot of our bound as a function of the attenuation. To produce it, we assumed that $R_{max} = 4R_{min}$, and R_{min} is given by the equation $2\alpha_0 R_{min}^{\beta_0}$ dB/km x 4 km $\simeq$ 10 $\log_{10}(A)$ dB. We used the absolute value of the greater of the left-hand-side and the right-hand-side of *(42)* (this explains the fact that the curve is not monotone: indeed, as the left-hand-side increases in absolute value, the right-hand-side decreases), It is clear from the figure that for larger attenuations, the relative difference between the ambiguous rain rates and our reference rain rate actually does not exceed 57%. It is important to keep in mind that this figure represents the extreme upper bound, not an r.m.s. value. In order to discuss "average" errors, one would need to introduce specific assumptions on the stochastic behavior of rain rates. This is beyond the scope of this paper. Figure 3 shows the upper bound on the relative error without any assumptions on the likelihood of one or another combination of rain parameters beyond the inequalities (25,26,35,36), or any probabilistic assumptions about the rain rate profile itself.

These bounds were obtained assuming that the rain is convective and may therefore vary over a relatively wide range $R_{min} \leq R \leq R_{max}$ -- indeed, we assumed that R_{max}/R_{min} = *4.* For stratiform rain, when $R_{max} \approx R_{min}$, (13) reduces to

$$R_1(r) = \left(\frac{\alpha_0}{\alpha_1}\right)^{1/\beta_1} R_0(r)^{\beta_0/\beta_1} \tag{43}$$

which implies that

$$0.7 R_0^{0.05} - 1 < \frac{R_1(r)}{R_0} - 1 \leq R_0^{0.05} - 1 \tag{44}$$

for any values of *A* small or large. These bounds show that the relative difference must remain small. Even when R_0 = 5 mm/hr, they implies that the extreme relative difference between R_1 and R_0 will not exceed 24 %. Thus, the fact that the rain rate does not vary much in range allows us to deduce a correspondingly tighter bound on the error.

In the genera] situation, if the path-integrated attenuation is not known exactly, i.e.

if p is not necessarily equal to 1 anymore, we still have the following version of (42)

$$\frac{0.7\, R_0(r)^{\lambda}}{\left(1 + * (1.15 p'' - 1)\right)^{0,94}} - 1 \leq \frac{R_1(r)}{R_0(r)} - 1 \leqq \frac{R_0(r)^{\lambda}}{\left(1 + \frac{A^m}{A^m - \rho^m}(\rho^m - 1.15)\right)^{0.94}} - 1 \quad (45)$$

where $m = \beta_1/b_1\ 20.74$, $|\lambda| = |\beta_0/\beta_1 - 1| < 0.05$, and where we are again assuming that the rain rates of interest fall within $R_{min} \leq R \leq R_{max}$ with $R_{max}/R_{min} = 4$. Figure 4 shows a contour plot of the upper bound for the error as given by (45), as a function of the one-way path-integrated attenuation and the error in its measurement. As long as the error remains below 0.5 dB, we can assert that the upper bound for the ambiguity will not exceed 100%. If the measurement error can reach 1 dB, the ambiguity will stay below 100% only if the one-way attenuation is at least 4 dB -- smaller attenuations will allow greater ambiguity, For measurement errors greater than 1.25 dB, we cannot expect the ambiguities to remain below 100%, regardless of the value of the path-integrated attenuation.

If we restrict our interest to a narrower range of possible rain rate values, say $R_{max}/R_{min} = 2$, the upper bound on the corresponding ambiguity is given by figure 5. The relative difference is manifestly smaller. For more nearly uniform rain, i.e. when $R_{max} \approx R_{min}$, our upper bound for the relative error is significantly lower. Figure 6 shows a contour plot of the upper bound for the ambiguity in this case. As long as the measurement error for the path-integrated attenuation remains below 1 dB, we can assert that the upper bound for the ambiguity will not exceed 70%. For a measurement error of less than 0.5 dB, the resulting ambiguity cannot exceed 40%.

Wc conclude this section with an example showing how much worse the ambiguity would be if we did not use coupled Z--R and k--R relations, For simplicity, we place ourselves in the situation of the previous paragraph and assume that R_0 is constant, say $R_0 = 20 mm/hr$, over the interval $0 \leq r \leq r_s = 4$ km. Let us also assume that the one-way path-integrated attenuation is known to within $\simeq \pm 0.2$ dB, i.e. that $0.9 \leq \rho \leq 1.1$. In this case, equation (9) becomes

$$R_1(r) = \frac{(\alpha_0/\alpha_1)^{1/\beta_1} R_0^{\beta_0/\beta_1}}{\left(1 + \frac{\rho^{\beta_1/b_1} - 1}{A^{\beta_1/b_1} - \rho^{\beta_1/b_1}}\, 10^{0.2 \frac{\beta_1}{b_1} \alpha_0 R_0^{\beta_0} r}\right)^{1/\beta_1}} \quad (46)$$

and constraint (11) becomes

$$\frac{a_1}{a_0} = \left(\frac{\alpha_1}{\alpha_0} \cdot \frac{A^{\beta_1/b_1} - 1}{A^{\beta_1/b_1} - \rho^{\beta_1/b_1}}\right)^{\beta_1/b_1} \cdot R_0^{b_0 - b_1\beta_0/\beta_1} \quad (47)$$

Notc that by directly integrating both sides of (46), one finds that

$$\int_0^{r_s} \alpha_1 R_1(r)^{\beta_1} dr = \alpha_0 R_0^{\beta_0} r_s - \frac{\log(p)}{0.2 \log(10)} \quad (48)$$

which in turn implies that

$$10^{-0.2 \quad s.''' \quad \alpha_1 R_1(r)^{\beta_1} dr} = 10^{-0.2\alpha_0 R_0^{\beta_0} r_s} \cdot \rho \tag{49}$$

as required. Wc use $\alpha_0 = 0,022$ and $\beta_0 = 1.1$ as reference values, so that the one-way attenuation is approximately 2.4 dB. From figure 6, the corresponding ambiguity using coupled Z–R and k–R relations cannot exceed 25%.

However, if we use $b_1 = 1.2$, $\alpha_1 = 0.034$, $\beta_1 = 1.2$ and $p = 1.1$, equation (46) implies that R_1 will be strictly decreasing, from $R_1(0) \simeq 10.4$ mm/hr to $R_1(4) \simeq 9.6$ mm/hr. The constraint (47) in this case states that a/a_o should equal 1.63.2060-11, which can be easily satisfied if $b_0 = 1,2$, $a_0 = 200$ and $al = 440$, for example. Similarly, if we use $b_1 = 1.2$, $\alpha_1 = 0.017$, $\beta_1 = 1$ and p = 0,9, equation (46) implies that R_1 will be strictly increasing, from $R_1(0) \simeq 37$ mm/hr to $R_1(4) \simeq 40.3$ mm/hr. This time, (47) states that al/a. should equal $0.687 \cdot 20^{b_0 - 1.32}$, which can be easily satisfied if $b_0 = 1.4$, $a. = 345$ and $al = 300$, for example, These simple examples illustrate how, without the coupling constraints on a, b, α and β, one can easily construct multiple solutions that differ from the reference by more than 100%, or over 4 times the maximum difference in the coupled case.

5 Conclusions

In summary, our use of coupled Z-R and k-R relations based on the drop-size distribution allows us to obtain upper bounds on the relative difference between the different rain rate profiles that can produce the same radar backscatter profile and the same path-integrated attenuation. At 13.8 GHz, the results indicate that the ambiguity between the multiple solutions is not prohibitively large. We considered the problem where the path-integrated attenuation figure available is exact and contains no error, and the problem where the attenuation was known only approximately. In each case, we obtained an upper bound on the difference between the mutually ambiguous multiple solutions to the problem.

When the integrated attenuation is known exactly, in the case of nearly uniform rain our upper bound for the relative difference between mutually ambiguous profiles is about 24%. In the case of convective rain, the upper bound decreases as the path-integrated attenuation increases, For 2 dB of one-way attenuation, the relative difference cannot exceed 70%. For 3 dB of one-way attenuation, the relative difference cannot exceed 62%. If the one-way attenuation is 4 dB or greater, the relative difference cannot exceed 57%.

On the other hand, when the path-integrated attenuation is known only approximately, the bound on the relative difference between the multiple solutions depends on the error in the attenuation figure. In the case of nearly uniform rain, if the attenuation error is less that 0.5 dB, we can conclude that the ambiguity will not exceed 40%, while if the attenuation error can reach 1 dB, wc. can only conclude that the ambiguity will remain below 70%. In the case of convective rain, for an error smaller than 0,5 dB the ambiguity will remain below 100%. For an error on the order of 1 dB, the ambiguity will remain below 100% only for sufficiently high attenuation, namely if the one-way attenuation is at least 4 dB. However if the error in the attenuation figure available is greater than or equal to 1.25 d13, the upper bound for the ambiguity will be greater than 100%.

These results have two major implications. First, the use of *coupled Z-R* and *k-R* relations does significantly reduce the ambiguities in the problem of retrieving a rain profile from a single-frequency radar backscatter profile together with the path-integrated attenuation. It is therefore important to use this coupling in implementing any inversion algorithms, especially when the algorithm in question proceeds by '(correcting" for the attenuation then inverting the '(corrected" reflectivities in two distinct steps. Second, if the error in the attenuation figure that is available exceeds 1 dB, then the additional constraint afforded by the path-integrated attenuation value is not sufficient to bring the ambiguity between the multiple rain-rate profile solutions below 100%. Figures 4, 5 and 6 show the detailed dependence of the ambiguity on the error and on the attenuation figure itself.

Last, the coupled Z-R and k-R relations which we have derived and used in the derivation of our bound on the ambiguity were obtained from T-matrix calculations using a very broad class of modeled drop-size distributions. Some of these drop-size distributions, corresponding to one or another possibly unrealistic combination of parameters, may never arise in nature. A careful analysis of a sufficiently large representative sampling of measured drop-size distributions should allow one to obtain coupled Z-R and k-R relations with sharper bounds on the parameters involved, thus producing correspondingly tighter bounds on the ambiguity. The implications for single-frequency radar systems such as TRMM are clear: the problem of obtaining physically realistic coupled Z-R and k-R relations with tight bounds on the parameters deserves serious attention, especially since, as we have shown, onc can then obtain correspondingly tight bounds on the differences between the mutually ambiguous rain rate profile solutions.

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Figure captions

Figure 1: Scatter diagram of k/R versus $\overline{D}$.

Figure 2: Scatter diagram of Z/R versus $\overline{D}$.

Figure 3: Graph of the upper bound on the ambiguity, in %, as a function of the one-way attenuation $(10 \log_{10} A)/2$ when there is no error in the measured attenuation.

Figure 4: Contour plot of the upper bound on the ambiguity, in %, as a function of the one-way attenuation $(10 \log_{10} A)/2$ and of the error in the measured attenuation $|10 \log_{10}(\rho)|$, with $R_{max}/R_{min} = 4$.

Figure 5: Contour plot of the upper bound on the ambiguity, in %, as a function of the one-way attenuation $(10 \log_{10} A)/2$ and of the error in the measured attenuation $|10 \log_{10}(\rho)|$, with $R_{max}/R_{min} = 2$.

Figure 6: Contour plot of the upper bound on the ambiguity, in %, as a function of the one-way attenuation $(10 \log_{10} A)/2$ and of the error in the measured attenuation $|10 \log_{10}(\rho)|$, with $R_{max}/R_{min} = 1$.

Appendix

l'here are two reasons a priori to suspect that the term $\epsilon(r)$ in equations (9) and (11) is relatively small and may therefore be ignored, First, it is proportional to ν, which is itself less than 0.1 . Second, its integrand is proportional to the derivative of R_0 with respect to range, divided by R_0, i.e. to the relative rate of increase of the rain rate with range (in mm/hr per km) -- a quantity which can physically not be very large. In fact, when R_0 is constant, its derivative is zero and therefore ϵ itself is identically zero.

Let us now rigorously verify that ϵ is indeed always negligibly small by making sure that it is significantly smaller than the larger of the other summands occuring in the same expression. Recall that

$$\epsilon(r) = \nu \int_0^r \frac{R_0'(r')}{R_0(r')} R_0(r')^\nu 10^{0.2\frac{\beta_1}{b_1}\int_{r'}^r \alpha_0 R_0(t)^{\beta_0} dt} dr' \quad (A,])$$

with v = $(\beta_1/b_1 - \beta_0/b_0)b_0$. Since $R_0(r')/R_0(0)$ cannot be greater than R_{max}/R_{min}, we can replace $R(r')$ in the integrand by $R_0(0)$. R_{max}/R_{min} outside the integral. Similarly, we replace $10^{0.2\frac{\beta_1}{b_1}\int_{r'}^r \alpha_0 R_0(t)^{\beta_0} dt}$ "in the integrand by its upper bound value when $r' = 0$. We then obtain

$$|\epsilon(r)| \leq |\nu| \left(\frac{R_{max}}{R_{min}}\right)^\nu \cdot R_0(0)^\nu 10^{0.2\frac{\beta_1}{b_1}\int_0^r \alpha_0 R_0(t)^{\beta_0} dt} \cdot \int_0^r \frac{R_0'(r')}{R_0(r')} dr' \quad (A.2)$$

For our reference value δ_0 = 0.15, we already know that ν is bounded by $|\nu| \leq 0.1$. Our inequality therefore becomes

$$|\epsilon(r)| \leq 0.1 \left(\frac{R_{max}}{R_{min}}\right)^{0.1} \cdot R_0(0)^\nu 10^{0.2\frac{\beta_1}{b_1}\int_0^r \alpha_0 R_0(t)^{\beta_0} dt} \cdot \log\left(\frac{R_{max}}{R_{min}}\right) \quad (A.3)$$

With R_{max}/R_{min} = 4, this bound becomes

$$|\epsilon(r)| \leq 0.16 R_0(0)^\nu 10^{0.2\frac{\beta_1}{b_1}\int_0^r \alpha_0 R_0(t)^{\beta_0} dt} \quad (A.4)$$

or

$$|\epsilon(r_s)| \leq 0.16\ R_0(0)^\nu A^{\beta_1/b_1} \quad (A.5)$$

when $r = r_s$. In other words, at the upper extreme, ϵ cannot exceed 16% of the absolute value of the larger of the other summands occuring in the same expressions in (9) and (11).

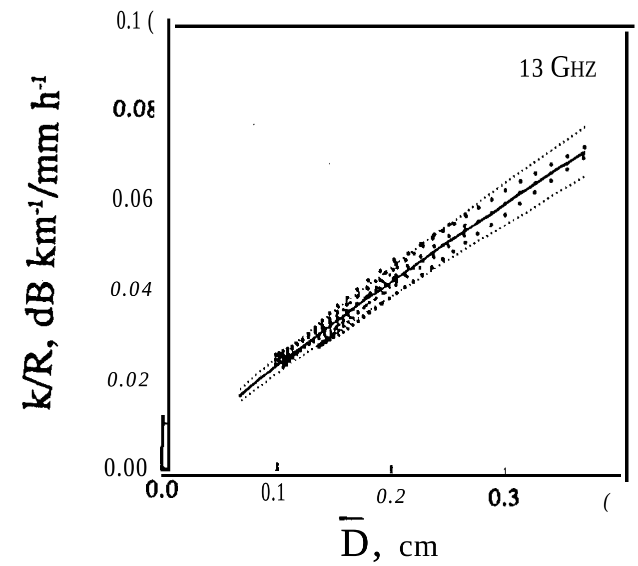


Figure 1

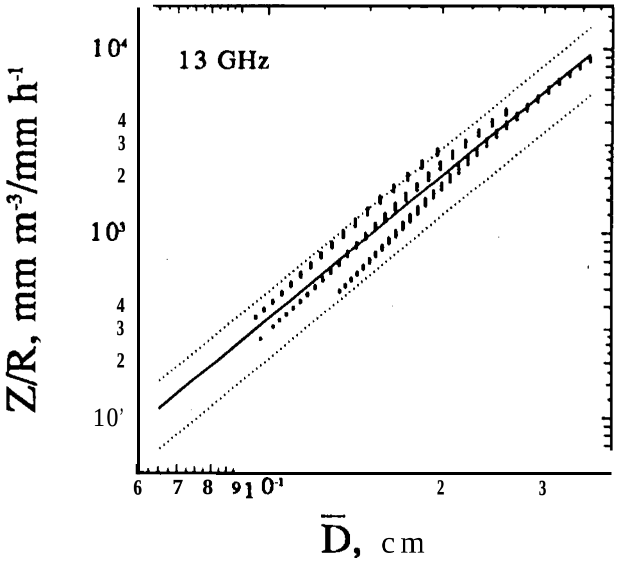
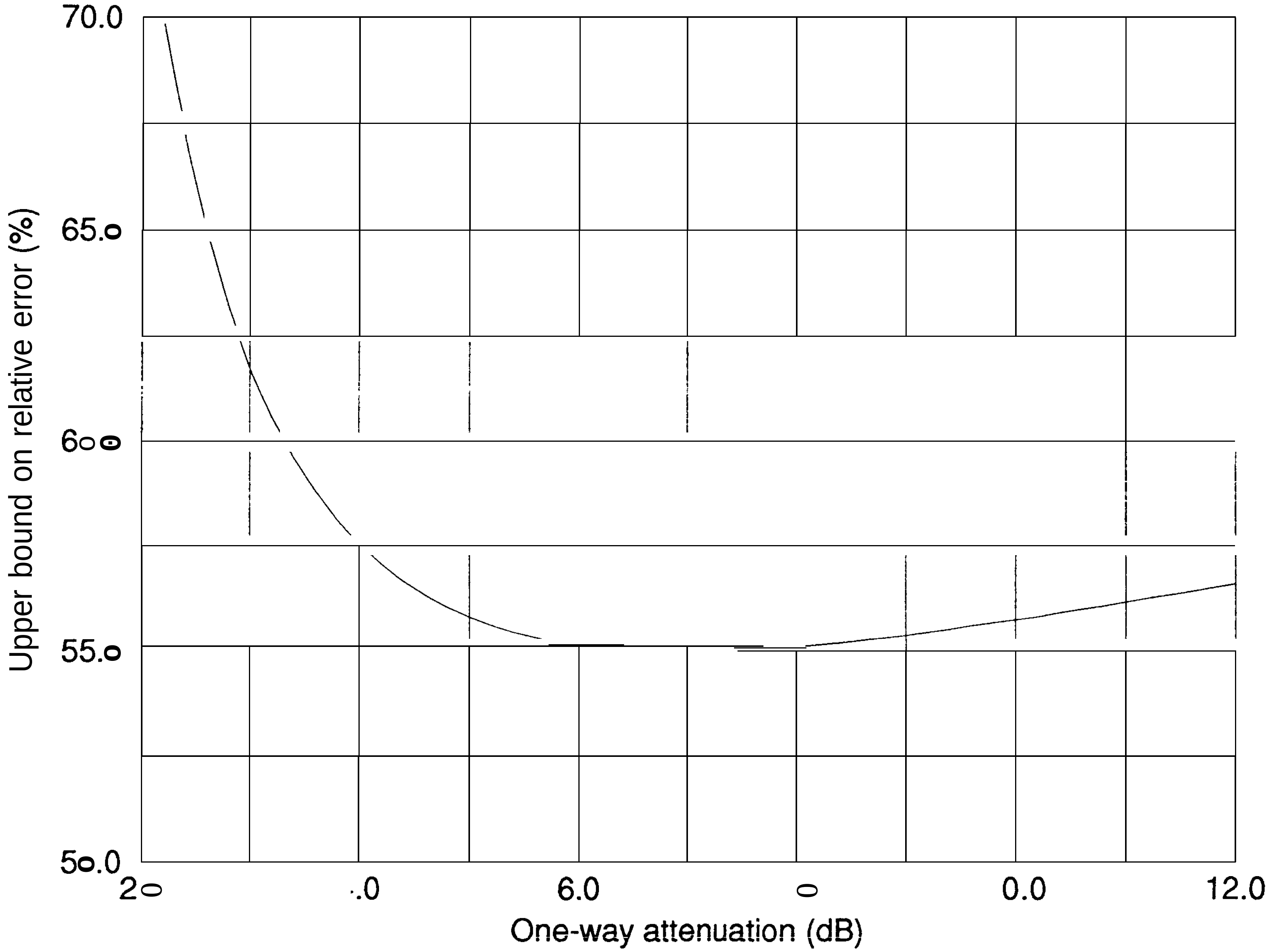


Figure 2

Figure 3


Upper bound on relative error (%)
70.0
65.0
6○0
55.0
50.0
2○
.0
6.0
0
0.0
12.0
One-way attenuation (dB)

Figu : 4 ~~Upper Bound on the ambiguity (%)~~

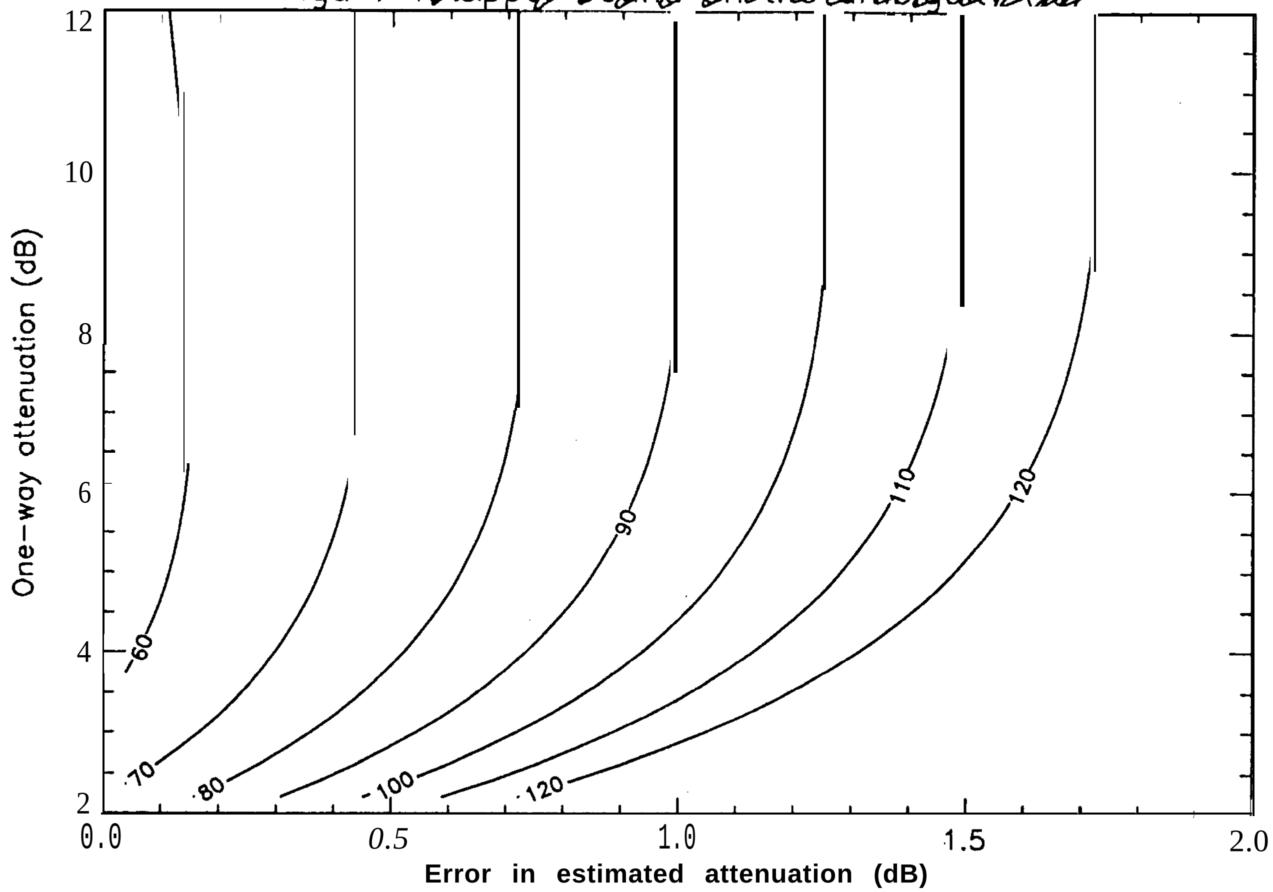
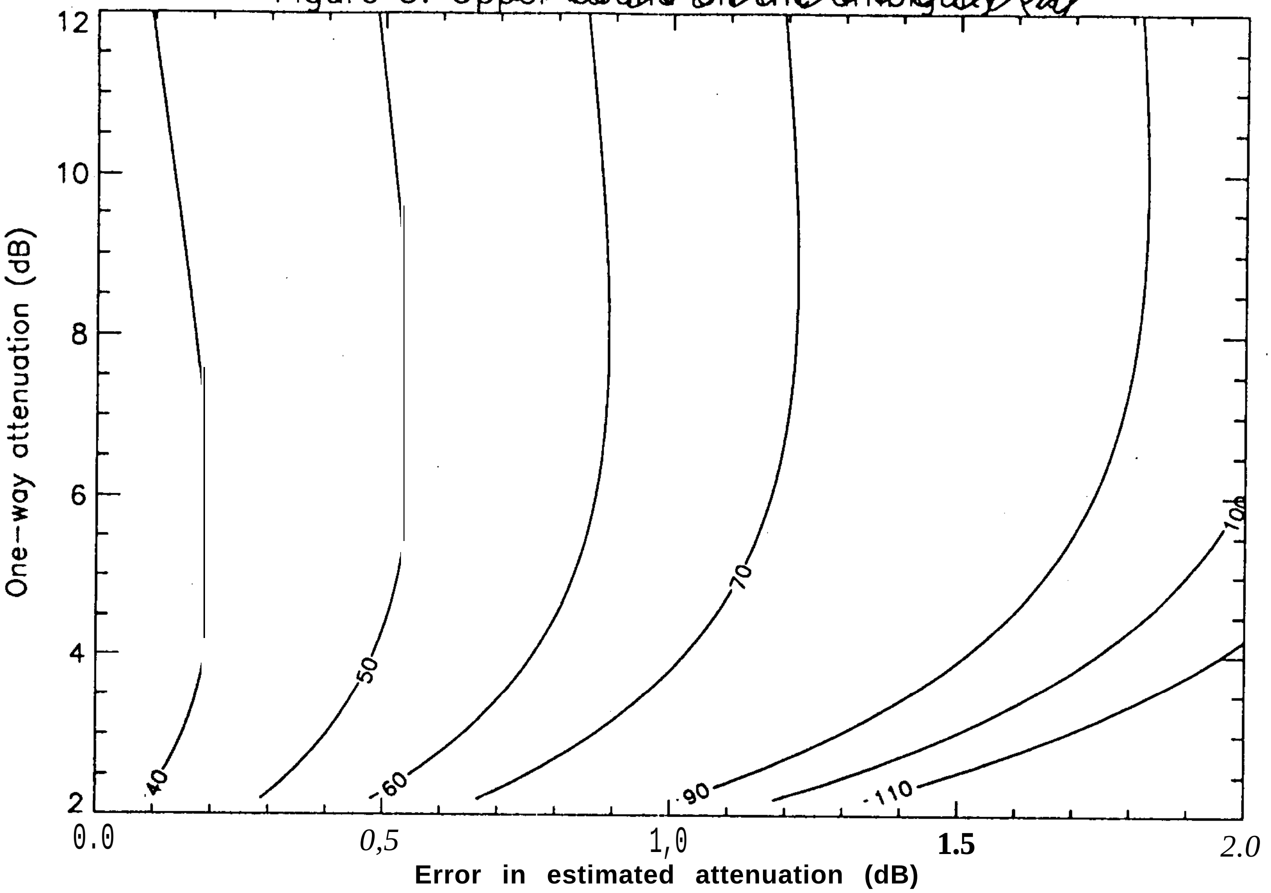
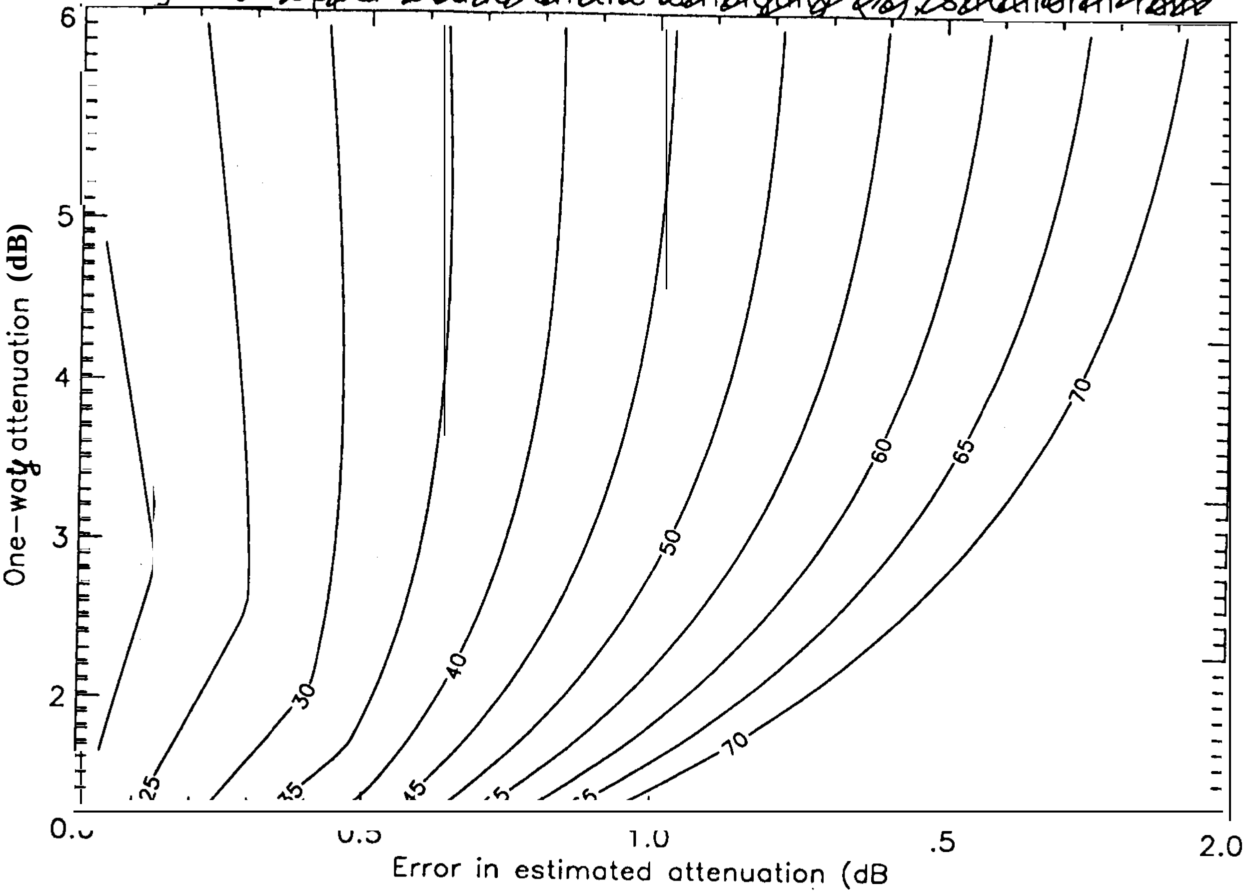


Figure 5: Upper bound on the ambiguity (%)





Figue 6: Upper bound on the ambiguity (%), stratiform rain